

# Three-loop $\Phi$ -derivable Approximation in QED

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In this paper we examine  $\Phi$ -derivable approximations in QED. General theorems tell us that the gauge dependence of the  $n$ -loop  $\Phi$ -derivable approximation shows up at order  $g^{2n}$  where  $g$  is the coupling constant. We consider the gauge dependence of the two-loop  $\Phi$ -derivable approximation to the Debye mass and show that it is of order  $e^4$  as expected. We solve the three-loop  $\Phi$ -derivable approximation in QED by expanding sum-integrals in powers of  $e^2$  and  $m/T$ , where  $m$  is the Debye mass which satisfies a variational gap equation. The results for the pressure and the Debye mass are accurate to order  $e^5$ .

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## I. INTRODUCTION

The thermodynamic functions for hot field theories can be calculated as a power series in the coupling constant  $g$  at weak coupling. The free energy has been calculated through to order  $g^4$  in [1, 2] for scalar  $\phi^4$  theory, in [3] for QED and in [2] for nonabelian gauge theories. The corresponding calculations to order  $g^5$  were carried out in Refs. [4, 5], Refs. [6, 7] and Refs. [8, 9], respectively. In Fig. 1, we show the successive perturbative approximations to  $\mathcal{P}/\mathcal{P}_{\text{ideal}}$  as a function of  $e(2\pi T)$ . Each partial sum is shown as a band obtained by varying the renormalization scale  $\mu$  by a factor of two around the central value  $\mu = 2\pi T$ . We have only done this variation for the  $e^5$  approximation. To express  $e(\mu)$  in terms of  $e(2\pi T)$ , we use the solution to the one-loop renormalization group equation in QED. The Figure shows that the weak-coupling expansion is poorly convergent unless the coupling constant is small and that it is very sensitive to the renormalization scale  $\mu$ . The lack of convergence seems to be related with screening and quasiparticles which is associated with the soft momentum scale of order  $eT$ . The instability of the weak-coupling expansion is a generic problem in hot field theories and makes it essentially useless for quantitative predictions.

There are several ways of systematically reorganizing the perturbative expansion to improve its convergence properties and various approaches have been discussed in detail in the review papers Refs. [10, 11, 12]. One of these methods is *screened perturbation theory* (SPT) which in

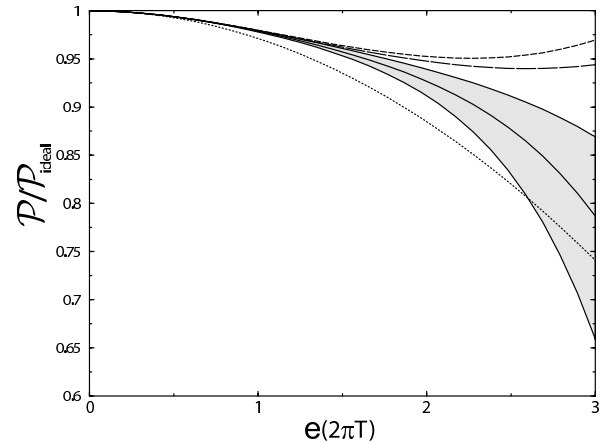


FIG. 1: Weak-coupling expansion for the pressure to orders  $e^2$  (dotted curve),  $e^3$  (dashed curve),  $e^4$  (long-dashed curve), and  $e^5$  (solid lines+band) normalized to that of an ideal gas as a function of  $e(2\pi T)$ .

the context of thermal field theory was introduced by Karsch, Patkós and Petreczky. [13] (See also refs. [14, 15, 16]). In this approach, one introduces a single variational parameter which has a simple interpretation as a thermal mass. In SPT a mass term is added to and subtracted from the scalar Lagrangian with the added piece kept as part of the free Lagrangian and the subtracted piece associated with the interactions. The mass parameter satisfies a variational equation which is obtained by the principle of minimal sensitivity.

In gauge theories, one cannot simply add and subtract a local mass term as this would violate gauge invariance. Instead one adds and subtracts to the Lagrangian a hard thermal loop (HTL) improvement term.

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The free part of the Lagrangian then includes the HTL self-energies and the remaining terms are treated as perturbations. *Hard thermal loop perturbation theory* is a manifestly gauge invariant approach that can be applied to static as well as dynamic quantities. SPT and HTL perturbation theory have been applied to three and two loops [17, 18, 19, 20, 21, 22], respectively, and the convergence properties are improved dramatically compared to the weak-coupling expansion.

The  $\Phi$ -derivable approach is another way of reorganizing the perturbative expansion which is variational of nature. In this approach, one uses the exact propagator as a variational function. Its formulation was first constructed by Luttinger and Ward [23] and by Baym [24]. Later it was generalized to relativistic field theories by Cornwall, Jackiw and Tomboulis [25]. The approach is based on the fact that the thermodynamic potential can be expressed in terms of the two-particle irreducible (2PI) effective action which has a diagrammatic expansion involving the 2PI skeleton graphs. Although here we focus on equilibrium physics we note that the 2PI formalism and its generalizations are also very useful when studying non-equilibrium real-time physics [26, 27, 28].

The  $\Phi$ -derivable approach has several attractive features. One is that it respects the global symmetries of the theory. Thus it is consistent with the conservation laws that follow from the Noether's theorem. Second, when evaluated at the stationary point, one is guaranteed thermodynamic consistency [24]. Finally, it turns out that the two-loop  $\Phi$ -derivable approximation has an additional property. The entropy reduces to the one-loop expression at the variational point. This property was first shown for QED by Vanderheyden and Baym [29] and later generalized to QCD by Blaizot, Iancu and Rebhan [30, 31, 32].

Applying the  $\Phi$ -derivable approach to quantum field theories, one is facing two nontrivial issues. The first issue is the question of renormalization. The three-loop calculations by Braaten and Petitgirard [34] for a massless scalar field theory indicate that there are ultraviolet divergences at order  $g^6$  that cannot be eliminated by any renormalization of the coupling constant. These calculations seem to contradict the results from the papers by van Hees and Knoll [35], and by Blaizot, Iancu and Reinosa [36], which show that the 2PI effective action can be systematically renormalized.

The second issue is that of gauge-fixing dependence. While the exact 2PI effective action is gauge independent at the stationary point, this property is often lost in approximations. The problem has recently been examined by Arrizabalaga and Smit [37], who showed that the  $n$ -loop  $\Phi$ -derivable approximation  $\Phi_n$ , which is defined by the truncation of the action functional after  $n$  loops, has a gauge dependence that shows up at order  $g^{2n}$ . Furthermore, if the  $n$ th order solution to the gap equation is used to evaluate the complete effective action, the gauge dependence first shows up at order  $g^{4n}$ . For a general

proof of the gauge invariance of the exact 2PI effective action we refer the reader to Ref. [38].

In gauge theories, approximate solutions to the gap equations in two-loop  $\Phi$ -derivable approximation in terms of HTL self-energies have been obtained by Blaizot, Iancu and Rebhan [30, 31, 32], and by Peshier [33]. For hard external momentum, these solutions are obtained by evaluating the one-loop self-energy diagrams with bare propagators. For soft external momentum, the solution is simply the HTL self-energies, which in the imaginary-time formalism reduces to the Debye mass. The resulting approximation reproduces the pressure to order  $g^3$ .

Finally, we mention that a dimensionally reduced version of the three-loop  $\Phi$ -derivable approximation was solved in the case of scalar field theory [12]. This was done by treating the contribution to the pressure from the nonzero Matsubara modes in strict perturbation theory and applying the  $\Phi$ -derivable approach to an effective three-dimensional field theory for the zero-frequency mode that has been obtained by dimensional reduction. The results are comparable to those obtained in SPT [17] and the  $\Phi$ -derivable approximation in 3+1 dimensions [34].

In this paper, we solve the three-loop  $\Phi$ -derivable approximation for QED. This is done by applying the strategy developed in Ref. [34]. It consists of expanding the sum-integrals systematically in powers of  $e$  and  $m/T$  where  $m$  is a variational mass parameter of order  $eT$ . Our result for the pressure reproduces the weak-coupling expansion to order  $g^5$  and is gauge invariant.

The paper is organized as follows. In Sec. II, we briefly discuss the application of the  $\Phi$ -derivable approach to QED and the general framework developed in Ref. [34] to solve it systematically. In Sec. III, we solve the two-loop  $\Phi$ -derivable approximation and discuss the issue of gauge dependence. In Sec. IV, we solve the three-loop  $\Phi$ -derivable approximation. We summarize and draw some conclusions in Sec. V. There are two appendices where our notation and conventions are given and where we list the sum-integrals and integrals that are needed.

## II. $\Phi$ -DERIVABLE APPROXIMATIONS

In this section, we briefly discuss the 2PI effective action formalism and  $\Phi$ -derivable approximations.

The Euclidean Lagrangian of massless QED is

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\gamma_\mu D_\mu\psi + \mathcal{L}_{\text{gf}}, \quad (1)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength tensor,  $D_\mu = \partial_\mu + ieA_\mu$  is the covariant derivative, and  $e$  is the electric coupling.  $\mathcal{L}_{\text{gf}}$  is the gauge-fixing part of the Lagrangian. In general covariant gauge, the gauge-fixing

part of the Lagrangian is

$$\mathcal{L}_{\text{gf}} = \frac{1}{2\xi} (\partial_\mu A_\mu)^2. \quad (2)$$

In the remainder of this Sec. and in Sec. III, we keep  $\xi$  general in order to discuss the problem of gauge dependence. In Sec. IV, we specialize to Feynman gauge ( $\xi = 1$ ), which by far is the easiest gauge for practical calculations.

The thermodynamic potential  $\Omega$  of QED is

$$\begin{aligned} \Omega[\Delta, S] = & \frac{1}{2} \text{Tr} \log \Delta^{-1} - \text{Tr} \log S^{-1} - \text{Tr} \log \Delta_{\text{gh}}^{-1} \\ & - \frac{1}{2} \text{Tr} \Pi \Delta + \text{Tr} \Sigma S + \text{Tr} \Pi_{\text{gh}} \Delta_{\text{gh}} \\ & + \Phi[\Delta, S], \end{aligned} \quad (3)$$

where  $\Delta_{\mu\nu}(P)$  and  $S(P)$  is the exact photon and electron propagator, respectively, and  $\Delta_{\text{gh}}(P)$  is the propagator for the ghost.  $\Pi_{\mu\nu}(P)$  is the polarization tensor and  $\Sigma(P)$  is the electron self-energy. We can then write

$$\Delta_{\mu\nu}^{-1}(P) = [\Delta_{\mu\nu}^0(P)]^{-1} + \Pi_{\mu\nu}(P), \quad (4)$$

$$S^{-1}(P) = \not{P} + \Sigma(P), \quad (5)$$

where  $\Delta_{\mu\nu}^0(P)$  is the free propagator in covariant gauge:

$$\Delta_{\mu\nu}^0(P) = \frac{\delta_{\mu\nu}}{P^2} - (1 - \xi) \frac{P_\mu P_\nu}{P^4}. \quad (6)$$

The trace in Eq. (3) is over Dirac and Lorentz indices as well as space-time. In covariant gauges, the ghost field decouples from the other fields and so the ghost self-energy  $\Pi_{\text{gh}}(P)$  vanishes identically<sup>1</sup>. The functional  $\Phi[\Delta, S]$  is the sum of all two-particle irreducible vacuum diagrams. We define the  $n$ -loop  $\Phi$ -derivable approximation  $\Omega_n$  to the thermodynamic potential  $\Omega$  as the truncation of the action functional after  $n$  loops. The two-particle irreducible vacuum diagrams are shown diagrammatically in Fig. 2 up to three-loop order. The corre-

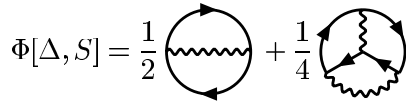


FIG. 2:  $\Phi$ -derivable two- and three-loop skeleton graphs.

sponding self-energies that are obtained by cutting a line, are shown in Figs. 3 and 4.

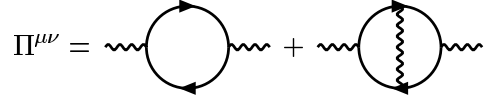


FIG. 3: One- and two-loop photon self-energy graphs.

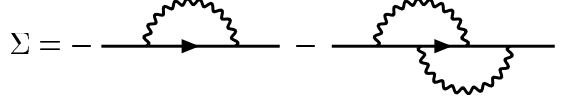


FIG. 4: One- and two-loop electron self-energy graphs.

The exact propagators satisfy the variational equations

$$\frac{\delta \Omega[\Delta, S]}{\delta \Delta} = 0, \quad (7)$$

$$\frac{\delta \Omega[\Delta, S]}{\delta S} = 0. \quad (8)$$

Using Eq. (3), the variational equations (7) and (8) can be written as

$$\Pi_{\mu\nu}(P) = 2 \frac{\delta \Phi[\Delta, S]}{\delta \Delta_{\mu\nu}(P)}, \quad (9)$$

$$\Sigma(P) = - \frac{\delta \Phi[\Delta, S]}{\delta S(P)}. \quad (10)$$

In QED, we know that thermal fluctuations generate a mass  $m$  for the zeroth component of the gauge field  $A_0$  which is of order  $eT$  and screens the interactions. The strategy for solving the  $n$ -loop  $\Phi$ -derivable approximation is to introduce a mass variable which is of order  $eT$  and then calculate the sum-integrals as double expansions in  $e^2$  and  $m/T$ . This strategy was developed in Ref. [34] in order to solve the three-loop  $\Phi$ -derivable approximation in scalar field theory. It turns out that the gap equations (9) and (10) have a recursive structure that allows us to solve for their dependence of the external momentum  $P$ . We follow Braaten and Petitgirard and choose the Debye mass as the mass parameter. The Debye mass is the solution to the equation

$$p^2 + \Pi_{00}(0, \mathbf{p}) = 0, \quad p^2 = -m^2. \quad (11)$$

In the variational equations (9) and (10), there are two important mass scales. One is *soft* and is of order  $eT$ . This scale is set by the Debye mass  $m$ . The other is the *hard* scale of order  $2\pi T$  and is set by the nonzero Matsubara modes. We will assume the coupling is sufficiently small so that the scales  $m$  and  $2\pi T$  are well separated. This allows one to expand the sum-integrals in powers of  $e^2$  and  $m/T$ . The gap equations will then be solved in the two momentum regions separately. For hard momentum  $P$ , we expand the polarization tensor as follows:

$$\begin{aligned} \Pi_{\mu\nu}(P) = & e^2 \Pi_{\mu\nu}^{2,0}(P) + e^4 [\Pi_{\mu\nu}^{4,0}(P) \\ & + \Pi_{\mu\nu}^{4,1}(P) + \dots] + \dots, \end{aligned} \quad (12)$$

<sup>1</sup> In nonabelian gauge theories, the ghost does not decouple in covariant gauges which makes the calculation significantly more involved. In Ref. [32] the authors are employing the temporal axial gauge in which the ghost does decouple. However, there are other problems with this gauge at finite temperature [39, 40].

where  $\Pi_{\mu\nu}^{n,k}(P)$  is of order  $T^2(m/T)^k$ . Similarly, the electron propagator is expanded as

$$\Sigma(P) = e^2 [\Sigma^{2,0}(P) + \Sigma^{2,1}(P) + \dots] + e^4 [\Sigma^{4,0}(P) + \Sigma^{4,1}(P) + \dots] + \dots, \quad (13)$$

where  $\Sigma^{n,k}(P)$  is of order  $T^2(m/T)^k$ . For soft momentum  $P = (0, \mathbf{p})$ , we expand the longitudinal part of polarization tensor as follows<sup>2</sup>:

$$\Pi_{00}(0, \mathbf{p}) = m^2 + e^2 [\sigma^{2,0}(p) + \sigma^{2,2}(p) + \dots] + e^4 [\sigma^{4,0}(p) + \dots] + \dots, \quad (14)$$

where  $\sigma^{n,k}(p)$  is of order  $m^2(m/T)^k$ .

For hard momentum, we can expand  $\frac{1}{2}\text{Tr} \log \Delta^{-1}$  about the free propagator, since the self-energy is perturbative corrections starting at order  $e^2$ . This yields

$$\begin{aligned} \frac{1}{2}\text{Tr} \log \Delta^{-1} &= \frac{1}{2}(d+1) \int_P \log P^2 + \frac{1}{2}e^2 \int_P \frac{\Pi_{\mu\mu}^{2,0}(P)}{P^2} \\ &+ \frac{1}{2}e^4 \int_P \left[ \frac{\Pi_{\mu\mu}^{4,0}(P)}{P^2} - \frac{1}{2} \frac{\Pi_{\mu\nu}^{2,0}(P)\Pi_{\mu\nu}^{2,0}(P)}{P^4} \right] + \dots \end{aligned} \quad (15)$$

The gauge-dependent terms in Eq. (15) drop out since the photon self-energy in QED is transverse to all orders:

$$P_\mu \Pi_{\mu\nu}(P) = 0. \quad (16)$$

For soft momentum, the expansion is

$$\begin{aligned} \frac{1}{2}\text{Tr} \log \Delta^{-1} &= \frac{1}{2}T \int_p \log(p^2 + m^2 + e^2 \sigma_{2,0}(p) + \dots) \\ &= \frac{1}{2}T \int_p \log(p^2 + m^2) + \frac{1}{2}e^2 T \int_p \frac{\sigma_{2,0}(p)}{p^2 + m^2} + \dots \end{aligned} \quad (17)$$

Again we have used the transversality of the photon propagator to eliminate the gauge-dependent terms. We do not need the expansion of  $\text{Tr} \Pi D$  since we will use the gap equation to eliminate this term.

We also need the expansions for  $\text{Tr} \log S^{-1}$  and  $\text{Tr} \Sigma S$ . Since the electron momentum is always hard, we can expand about the free propagator and obtain

$$\begin{aligned} \text{Tr} \log S^{-1} &= 2 \int_{\{P\}} \log P^2 \\ &+ e^2 \int_{\{P\}} \text{Tr} \left[ \frac{\Sigma^{2,0}(P) \not{P}}{P^2} + \frac{\Sigma^{2,1}(P) \not{P}}{P^2} + \dots \right] \\ &- \frac{1}{2}e^4 \int_{\{P\}} \left[ \frac{\Sigma^{2,0}(P) \not{P} \Sigma^{2,0}(P) \not{P}}{P^4} + \dots \right] + \dots \end{aligned} \quad (18)$$

$$\begin{aligned} \text{Tr} \Sigma S &= e^2 \int_{\{P\}} \text{Tr} \left[ \frac{\Sigma^{2,0}(P) \not{P}}{P^2} + \frac{\Sigma^{2,1}(P) \not{P}}{P^2} + \dots \right] \\ &- e^4 \int_{\{P\}} \left[ \frac{\Sigma^{2,0}(P) \not{P} \Sigma^{2,0}(P) \not{P}}{P^4} + \dots \right] + \dots \end{aligned} \quad (19)$$

<sup>2</sup> Since the infrared limit of the other components of  $\Pi_{\mu\nu}(0, \mathbf{p})$  vanishes, the corresponding contribution to the free energy also vanishes.

where the trace on the right-hand side is only over Dirac indices.

The contribution from the ghost field is as usual

$$\text{Tr} \log \Delta_{\text{gh}}^{-1} = \int_P \log P^2. \quad (20)$$

By inserting the expansions for the self-energies into the gap equations and expanding systematically in powers of  $e$  and  $m/T$ , we obtain expressions for  $\Pi_{\mu\nu}^{n,k}(P)$ ,  $\Sigma^{n,k}(P)$  and  $\sigma_{\mu\nu}^{n,k}(p)$ . By matching coefficients of  $e^n$  on both sides and solving the equations simultaneously and recursively.

### III. TWO LOOPS

In the two-loop  $\Phi$ -derivable approximation, there is only a single diagram contributing to  $\Phi[D, S]$  which is the left diagram in Fig. 2. The two-loop thermodynamic potential  $\Omega_2$  is

$$\begin{aligned} \Omega_2[\Delta, S] &= \frac{1}{2}\text{Tr} \log \Delta^{-1} - \text{Tr} \log S^{-1} - \text{Tr} \log \Delta_{\text{gh}}^{-1} \\ &- \frac{1}{2}\text{Tr} \Pi \Delta + \text{Tr} \Sigma S + \frac{1}{2}e^2 \int_{P\{Q\}} \\ &\times \text{Tr} [S(Q) \gamma^\mu S(P+Q) \gamma^\nu \Delta_{\mu\nu}(P)] \end{aligned} \quad (21)$$

The gap equations are obtained by varying the thermodynamic potential  $\Omega_2$  with respect to  $\Pi_{\mu\nu}(P)$  and  $\Sigma(P)$ :

$$\Pi_{\mu\nu}(P) = e^2 \int_{\{Q\}} \text{Tr} [S(Q) \gamma^\mu S(P+Q) \gamma^\nu], \quad (22)$$

$$\Sigma(P) = e^2 \int_Q \gamma^\mu S(P+Q) \gamma^\nu \Delta_{\mu\nu}(Q). \quad (23)$$

It follows from the coupled gap equations that both  $\Pi_{\mu\nu}(P)$  and  $\Sigma(P)$  are nontrivial functions of the external momentum  $P$ .

The gap equation (22) can be used to simplify equation (21) for  $\Omega_2$ :

$$\begin{aligned} \Omega_2 &= \frac{1}{2}\text{Tr} \log \Delta^{-1} - \text{Tr} \log \Delta_{\text{gh}}^{-1} - \text{Tr} \log S^{-1} \\ &+ \text{Tr} \Sigma S. \end{aligned} \quad (24)$$

Substituting the expansions for the various terms into (24) and truncating at the appropriate order, we obtain

$$\begin{aligned} \Omega_2 &= \frac{1}{2}(d-1) \int_P \log P^2 + \frac{1}{2}T \int_p \log(p^2 + m^2) \\ &- 2 \int_{\{P\}} \log P^2 + \frac{1}{2}e^2 \int_P \frac{\Pi_{\mu\mu}^{2,0}(P)}{P^2}. \end{aligned} \quad (25)$$

We note that the function  $\Sigma^{2,0}(P)$  drops out. In the two-loop  $\Phi$ -derivable approximation, we only need the trace

of  $\Pi_{\mu\nu}^{2,0}(P)$ , while in the three-loop  $\Phi$ -derivable approximation, we need the function itself.

The solution to the gap equations for hard momentum to order  $e^2$  are obtained by using bare propagators in the loops. In this manner, we find

$$\Pi_{\mu\nu}^{2,0}(P) = \oint_{\{Q\}} \left[ \frac{8Q_\mu Q_\nu}{Q^2(P+Q)^2} - \frac{4\delta_{\mu\nu}}{Q^2} + \frac{2P^2\delta_{\mu\nu}}{Q^2(P+Q)^2} + \frac{4P_\mu Q_\nu + 4P_\nu Q_\mu}{Q^2(P+Q)^2} \right]. \quad (26)$$

We also need to solve the gap equation for soft momentum in order to determine the Debye mass. Through order  $e^3$ , the longitudinal part of polarization tensor at zero frequency reads

$$\Pi_{00}(0, \mathbf{p}) = m^2, \quad (27)$$

where the Debye mass is

$$\begin{aligned} m^2 &= -4(d-1)e^2 \oint_{\{Q\}} \frac{1}{Q^2} \\ &= \frac{16\pi^2}{3} \alpha T^2, \end{aligned} \quad (28)$$

where  $\alpha = e^2/(4\pi)^2$ . The thermodynamic potential through order  $e^3$  then reduces to

$$\begin{aligned} \Omega_2 &= \frac{1}{2}(d-1) \oint_P \log P^2 + \frac{1}{2}T \int_p \log(p^2 + m^2) \\ &\quad - 2 \oint_{\{P\}} \log P^2 - (d-1)e^2 \\ &\quad \times \left[ \oint_{P\{Q\}} \frac{2}{P^2 Q^2} - \oint_{\{PQ\}} \frac{1}{P^2 Q^2} \right]. \end{aligned} \quad (29)$$

Using the expressions for the integrals and sum-integrals in the appendices, this reduces to

$$\Omega_2 = -\frac{11\pi^2 T^4}{180} \left[ 1 - \frac{50}{11}\alpha + \frac{320\sqrt{3}}{33}\alpha^{3/2} \right]. \quad (30)$$

Eq. (30) agrees with the weak-coupling result through order  $e^3$  [6]. Thus the two-loop  $\Phi$ -derivable approximation sums up the leading contribution from the plasmon diagrams.

We close this section by discussing the problem of gauge dependence that arises when going beyond order  $e^3$ . For example, to calculate the Debye mass to order  $e^4$ , we need to include the function  $\Sigma^{2,0}(P)$  in the dressed electron propagator on the right-hand side of the gap equation (22). This function is

$$\begin{aligned} \Sigma^{2,0}(P) &= (1-d) \oint_Q \frac{(P+Q)}{Q^2(P+Q)^2} - (1-\xi) \\ &\quad \times \oint_Q \left[ \frac{Q}{Q^4} - \frac{P}{Q^2(P+Q)^2} - \frac{P^2 Q}{Q^4(P+Q)^2} \right]. \end{aligned} \quad (31)$$

The function  $\Sigma^{2,0}(P)$  arises from hard photon momenta in the self-energy graph in Fig. 4. We note that it is gauge dependent and this is due the photon line in the one-loop self-energy graph shown in Fig. 4. Since  $\Sigma^{2,0}(P)$  is gauge dependent, this introduces a gauge dependence at order  $e^4$  in the longitudinal part of polarization tensor. At zero frequency, one finds:

$$\begin{aligned} \Pi_{00}(0, \mathbf{p}) &= -4(d-1)e^2 \oint_{\{Q\}} \frac{1}{Q^2} \\ &\quad + \frac{2}{3}(d-1)e^2 p^2 \oint_{\{Q\}} \frac{1}{Q^4} \\ &\quad + \frac{8}{3}(d-1)(d-3)e^4 \oint_{\{Q\}} \frac{1}{Q^4} \left[ \oint_R \frac{1}{R^2} - \oint_{\{R\}} \frac{1}{R^2} \right] \\ &\quad + 8(1-\xi)e^4 \oint_{\{PQ\}} \left[ \frac{P_0^2}{P^2 Q^4(P+Q)^2} - \frac{2P_0^2}{P^4 Q^4} \right]. \end{aligned} \quad (32)$$

The equation for  $\Pi_{00}(0, \mathbf{p})$  is ultraviolet divergent and requires renormalization. The divergence proportional to  $p^2$  is removed by wave-function renormalization in the usual manner (see also Sec. IV). The other divergence which arises from the last line in Eq. (32) can only be removed by a gauge-dependent renormalization of the coupling constant<sup>3</sup>. This would lead to a gauge-dependent gap equation and Debye mass. Eq. (32) depends on the gauge-fixing parameter, but this is not in contradiction with the fact that the photon self-energy is manifestly gauge invariant. The point is that in the two-loop  $\Phi$ -derivable approximation, we are not including all contributions to  $\Pi_{\mu\nu}(P)$  of order  $e^4$ . This is done when one considers the three-loop approximation and we have explicitly checked that the gauge dependence cancels algebraically as we include the two-loop self-energy graph in Fig. 3.

#### IV. THREE LOOPS

The three-loop  $\Phi$ -derivable approximation to the free energy is

$$\begin{aligned} \Omega_3[\Delta, S] &= \frac{1}{2} \text{Tr} \log \Delta^{-1} - \text{Tr} \log S^{-1} - \text{Tr} \log \Delta_{\text{gh}}^{-1} \\ &\quad - \frac{1}{2} \text{Tr} \Pi \Delta + \text{Tr} \Sigma S \\ &\quad + \frac{1}{2} e^2 \oint_{P\{Q\}} \text{Tr} [S(Q) \gamma^\mu S(P+Q) \gamma^\nu \Delta_{\mu\nu}(P)] \\ &\quad + \frac{1}{4} e^4 \oint_{P\{QR\}} \text{Tr} [S(Q) \gamma^\mu S(R) \gamma^\alpha S(R-P) \end{aligned}$$

<sup>3</sup> Only in Feynman gauge is the polarization tensor finite after wave-function renormalization.

$$\times \gamma^\nu S(Q-P) \gamma^\beta \Delta_{\mu\nu}(P) \Delta_{\alpha\beta}(Q-R)] . \quad (33)$$

The gap equations are again obtained by varying (33) with respect to the photon and electron self-energies:

$$\begin{aligned} \Pi_{\mu\nu}(P) &= e^2 \not\!\!\int_{\{Q\}} \text{Tr} [S(Q) \gamma^\mu S(P+Q) \gamma^\nu] , \\ &+ e^4 \not\!\!\int_{\{QR\}} \text{Tr} [S(Q) \gamma^\mu S(R) \gamma^\alpha \\ &\times S(R-P) \gamma^\nu S(Q-P) \gamma^\beta] \Delta_{\alpha\beta}(Q-R) , \end{aligned} \quad (34)$$

$$\begin{aligned} \Sigma(P) &= e^2 \not\!\!\int_Q \gamma^\mu S(P+Q) \gamma^\nu \Delta_{\mu\nu}(Q) \\ &+ e^4 \not\!\!\int_{Q\{R\}} \gamma^\mu S(R) \gamma^\alpha S(R-Q) \gamma^\nu \\ &\times S(P-Q) \gamma^\beta \Delta_{\alpha\beta}(Q) \Delta_{\mu\nu}(R-P) . \end{aligned} \quad (35)$$

The gap equation for  $\Pi_{\mu\nu}(P)$  can be used to simplify the expression for the thermodynamic potential:

$$\begin{aligned} \Omega_3[\Delta, S] &= \frac{1}{2} \text{Tr} \log \Delta^{-1} - \text{Tr} \log S^{-1} - \text{Tr} \log \Delta_{\text{gh}}^{-1} \\ &+ \text{Tr} \Sigma S \\ &- \frac{1}{4} e^4 \not\!\!\int_{P\{QR\}} \text{Tr} [S(Q) \gamma^\mu S(R) \gamma^\alpha S(R-P) \\ &\times \gamma^\nu S(Q-P) \gamma^\beta \Delta_{\mu\nu}(P) \Delta_{\alpha\beta}(Q-R)] . \end{aligned} \quad (36)$$

Substituting the expansions (15)–(19) into (36), we obtain

$$\begin{aligned} \Omega_3 &= \frac{1}{2}(d-1) \not\!\!\int_P \log P^2 + \frac{1}{2} T \int_p \log(p^2 + m^2) \\ &+ \frac{1}{2} e^2 T \int_p \frac{\sigma^{2,0}(p)}{p^2 + m^2} - 2 \not\!\!\int_{\{P\}} \log P^2 \\ &+ \frac{1}{2} e^2 \not\!\!\int_P \frac{\Pi_{\mu\mu}^{2,0}(P)}{P^2} \\ &+ \frac{1}{2} e^4 \not\!\!\int_P \left[ \frac{\Pi_{\mu\mu}^{4,0}(P)}{P^2} + \frac{\Pi_{\mu\mu}^{4,1}(P)}{P^2} \right. \\ &\quad \left. - \frac{1}{2} \not\!\!\int_P \frac{\Pi_{\mu\nu}^{2,0}(P) \Pi_{\mu\nu}^{2,0}(P)}{P^4} \right] \\ &- \frac{1}{2} e^4 \not\!\!\int_{\{P\}} \text{Tr} \left[ \frac{\Sigma^{2,0}(P) \not\!P \Sigma^{2,0}(P) \not\!P}{P^4} \right. \\ &\quad \left. + 2 \frac{\Sigma^{2,0}(P) \not\!P \Sigma^{2,1}(P) \not\!P}{P^4} + \frac{\Sigma^{2,1}(P) \not\!P \Sigma^{2,1}(P) \not\!P}{P^4} \right] \\ &+ \frac{1}{2} (d-1)(5-d) e^4 \not\!\!\int_{\{PQR\}} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} \\ &+ (d-1)(d-3) e^4 \not\!\!\int_{PQ\{R\}} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} \\ &+ 8(d-1) e^4 \not\!\!\int_{\{Q\}R} \frac{Q_0 R_0}{Q^4 R^2 (Q+R)^2} \left( T \int_p \frac{1}{p^2 + m^2} \right) . \end{aligned} \quad (37)$$

In the three-loop  $\Phi$ -derivable approximation, we only need the trace of the functions  $\Pi_{\mu\nu}^{4,0}(P)$  and  $\Pi_{\mu\nu}^{4,1}(P)$ . These traces are significantly simpler to calculate than the functions themselves. Furthermore, it turns out that the order- $e^4$  term in equation (35) for the electron self-energy first contributes to the pressure at order  $e^7$  and so it is not needed. In addition to the functions listed in Sec. III, we need the following functions for hard  $P$

$$\Sigma^{2,1}(P) = \frac{2p_0 \gamma_0 - \not\!P}{P^2} T \int_q \frac{1}{q^2 + m^2} , \quad (38)$$

$$\begin{aligned} \Pi_{\mu\mu}^{4,0}(P) &= -4(d-1)^2 \not\!\!\int_{\{Q\}R} \left[ \frac{1}{Q^4 R^2} - \frac{P^2}{Q^4 R^2 (Q+R)^2} \right. \\ &\quad \left. + \frac{1}{R^2 (P+Q)^2 (Q+R)^2} - \frac{2P \cdot R}{Q^2 R^2 (P+Q)^2 (Q+R)^2} \right] \\ &- 4(d-1)^2 \not\!\!\int_{\{QR\}} \left[ \frac{P^2}{Q^4 R^2 (P+Q)^2} - \frac{1}{Q^4 R^2} \right] \\ &- 4(d-1) \not\!\!\int_{\{QR\}} \left[ \frac{4(Q \cdot R)^2}{Q^2 R^2 (R-P)^2 (Q-P)^2 (Q-R)^2} \right. \\ &\quad \left. + \frac{P^4}{Q^2 R^2 (R-P)^2 (Q-P)^2 (Q-R)^2} \right. \\ &\quad \left. - \frac{4P^2}{Q^2 R^2 (Q-P)^2 (Q-R)^2} \right. \\ &\quad \left. - \frac{d-7}{2} \frac{P^2}{Q^2 R^2 (R-P)^2 (Q-P)^2} \right. \\ &\quad \left. + (d-3) \frac{1}{R^2 (Q-P)^2 (Q-R)^2} \right. \\ &\quad \left. - \frac{2Q^2}{R^2 (R-P)^2 (Q-P)^2 (Q-R)^2} \right. \\ &\quad \left. + \frac{2}{R^2 (R-P)^2 (Q-R)^2} \right] , \end{aligned} \quad (39)$$

$$\begin{aligned} \Pi_{\mu\mu}^{4,1}(P) &= 2(d-1) \not\!\!\int_{\{Q\}} \left[ \frac{2P^2}{Q^4 (P+Q)^2} - \frac{2}{Q^4} \right. \\ &\quad \left. + \frac{8q_0^2}{Q^6} - \frac{8P^2 q_0^2}{Q^6 (P+Q)^2} - \frac{4p_0 q_0 P^2}{Q^4 (P+Q)^4} \right. \\ &\quad \left. - \frac{4P^2 q_0^2}{Q^4 (P+Q)^4} \right] \left( T \int_r \frac{1}{r^2 + m^2} \right) . \end{aligned} \quad (40)$$

The functions  $\Sigma^{2,1}(P)$  and  $\Pi_{\mu\nu}^{4,1}(P)$  arise when the photon momentum in the relevant Feynman diagram is soft, while  $\Pi_{\mu\nu}^{4,0}(P)$  is when the photon momentum is hard.

Through order  $e^5$ , the longitudinal part of polarization tensor at zero frequency can be written as

$$\begin{aligned} \Pi_{00}(0, \mathbf{p}) &= -4(d-1) e^2 \not\!\!\int_{\{Q\}} \frac{1}{Q^2} \\ &\quad + \frac{2}{3} (d-1) e^2 p^2 \not\!\!\int_{\{Q\}} \frac{1}{Q^4} \end{aligned}$$

$$\begin{aligned}
& +4(d-1)(d-3)e^4 \oint_{\{Q\}} \frac{1}{Q^4} \left[ \oint_R \frac{1}{R^2} - \oint_{\{R\}} \frac{1}{R^2} \right] \\
& +4(d-1)(d-3)e^4 \oint_{\{P\}} \frac{1}{P^4} \left( T \int_r \frac{1}{r^2 + m^2} \right). \quad (41)
\end{aligned}$$

We have explicitly checked that (41) is independent of the gauge-fixing parameter  $\xi$  in contrast to the gauge-dependent expression (32). Eq (41) can be rewritten as

$$\Pi_{00}(0, \mathbf{p}) = m^2 + e^2 \sigma^{2,0}(p), \quad (42)$$

where the Debye mass satisfies

$$\begin{aligned}
m^2 &= -4(d-1)e^2 \oint_{\{Q\}} \frac{1}{Q^2} \\
&+ \frac{2}{3}(d-1)e^2 p^2 \oint_{\{Q\}} \frac{1}{Q^4} \Big|_{p^2=-m^2} \\
&+4(d-1)(d-3)e^4 \oint_{\{Q\}} \frac{1}{Q^4} \left[ \oint_R \frac{1}{R^2} - \oint_{\{R\}} \frac{1}{R^2} \right] \\
&+4(d-1)(d-3)e^4 \oint_{\{P\}} \frac{1}{P^4} \left( T \int_r \frac{1}{r^2 + m^2} \right), \quad (43)
\end{aligned}$$

and

$$\sigma^{2,0}(p) = \frac{2}{3}(d-1)e^2(p^2 + m^2) \oint_{\{Q\}} \frac{1}{Q^4}. \quad (44)$$

The gap equation (43) is ultraviolet divergent and requires renormalization. The divergence is proportional to  $p^2$  and is removed by wave-function renormalization:

$$Z_A = 1 - \frac{e^2}{12\pi^2\epsilon}. \quad (45)$$

After having renormalized the static polarization tensor  $\Pi_{00}(0, \mathbf{p})$ , the gap equation (11) reduces to

$$\begin{aligned}
m^2 &= \frac{16\pi^2}{3} T^2 \alpha \left\{ 1 - \frac{8}{3} \left[ \log \frac{\mu}{4\pi T} + \gamma + 2 \log 2 + \frac{7}{4} \right. \right. \\
&\quad \left. \left. - 18 \left( \frac{m}{4\pi T} \right) \right] \alpha \right\}. \quad (46)
\end{aligned}$$

The result for the Debye mass (46) agrees with the weak-coupling result [7, 41] through order  $e^5$ .

Inserting the expressions (38)–(40) and (44) into (37), the three-loop  $\Phi$ -derivable approximation then becomes

$$\begin{aligned}
\Omega_3 &= \frac{1}{2}(d-1) \oint_P \log P^2 + \frac{1}{2} T \int_p \log(p^2 + m^2) \\
&- 2 \oint_{\{P\}} \log P^2 - e^2(d-1) \\
&\times \left[ \oint_{P\{Q\}} \frac{2}{P^2 Q^2} - \oint_{\{PQ\}} \frac{1}{P^2 Q^2} \right]
\end{aligned}$$

$$\begin{aligned}
& + (d-1)^2 e^4 \oint_{\{P\}} \frac{1}{P^4} \left[ \oint_{\{Q\}} \frac{1}{Q^2} - \oint_Q \frac{1}{Q^2} \right]^2 \\
& - 4(d-3)e^4 \oint_{P\{QR\}} \frac{1}{P^4 Q^2 R^2} \\
& - 2(d-1)e^4 \oint_{PQ\{R\}} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} \\
& + \frac{1}{2}(d^2 - 8d + 11)e^4 \\
& \times \oint_{\{PQR\}} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} \\
& - 2(d-1)^2 e^4 \oint_{\{P\}QR} \frac{QR}{P^2 Q^2 R^2 (P+Q)^2 (P+R)^2} \\
& - 16e^4 \oint_{P\{QR\}} \frac{(QR)^2}{P^4 Q^2 R^2 (P+Q)^2 (P+R)^2} \\
& - (d-1)(d-3)e^4 \oint_{\{P\}} \frac{1}{P^4} \left( T \int_q \frac{1}{q^2 + m^2} \right)^2. \quad (47)
\end{aligned}$$

Note that we have kept a term that is proportional to  $e^4 m^2$  and first contributes at order  $e^6$ . This term arises from three-loop diagrams where both photons are soft<sup>4</sup>. This contribution is manifestly gauge invariant and we include this selective resummation in our final result. It is interesting to note that the only contribution at order  $e^5$  comes from the Debye mass; all the other contributions cancel algebraically

The expression for  $\Omega_3$  is ultraviolet divergent. The divergences can be eliminated by renormalizing the coupling constant. This is done by the substitution  $e^2 \rightarrow Z_e^2 e^2$ , where

$$Z_e^2 = 1 + \frac{e^2}{12\pi^2\epsilon}. \quad (48)$$

Using the expressions for the sum-integrals and integrals listed in the appendices, we obtain

$$\begin{aligned}
\Omega_3 &= -\frac{11\pi^2 T^4}{180} \left\{ 1 - \frac{50}{11}\alpha + \frac{960}{11} \left( \frac{m}{4\pi T} \right)^3 \right. \\
&- \left[ \frac{400}{33} \left( \log \frac{\mu}{4\pi T} + \frac{3}{5}\gamma - \frac{2}{5} \frac{\zeta'(-3)}{\zeta(-3)} \right. \right. \\
&\quad \left. \left. + \frac{4}{5} \frac{\zeta'(-1)}{\zeta(-1)} + \frac{319}{80} - \frac{156}{25} \log 2 \right) \right. \\
&\quad \left. \left. + \frac{11520}{11} \left( \frac{m}{4\pi T} \right)^2 \right] \alpha^2 \right\}. \quad (49)
\end{aligned}$$

Using the expression for the Debye mass in Eq. (46), one can show that the three-loop  $\Phi$ -derivable approximation

<sup>4</sup> The complete  $e^4 m^2$  contribution can also be obtained in a two-loop calculation using the dimensionally reduced theory of QED (electrostatic QED) derived in [7, 42].

agrees with the weak-coupling expansion through order  $e^5$  [6, 7].

The renormalization group equation that follows from (48) is

$$\mu \frac{de^2}{d\mu} = \frac{e^4}{6\pi^2}, \quad (50)$$

which coincides with the standard one-loop running of the coupling.

In Fig. 5, we show the two and three-loop  $\Phi$ -derivable approximations to the pressure normalized to that of an ideal gas shown as dashed and solid lines, respectively. In the three-loop approximation, the band is obtained in the usual manner by varying the renormalization scale  $\mu$ . The three-loop band is slightly narrower when compared to the  $e^5$ -band in Fig. 1. The approximations also seem to be slightly more stable than the successive weak-coupling approximations. However, the final result does not seem to be a dramatic improvement over the  $e^5$  result.

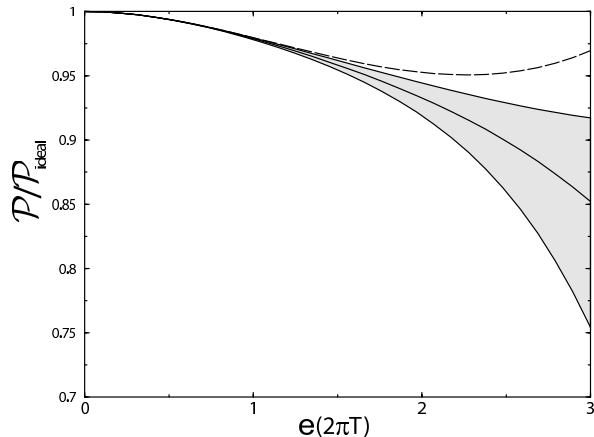


FIG. 5: Two- and three-loop  $\Phi$ -derivable approximations to the pressure normalized to that of an ideal gas as a function of  $e(2\pi T)$  shown as dashed and solid lines, respectively. The three-loop band is obtained by varying the renormalization scale,  $\mu$ , by a factor of two around the central value  $\mu = 2\pi T$ . Note that the scale is different than in Fig. 1.

## V. SUMMARY

In this paper we have solved the three-loop  $\Phi$ -derivable approximation for the Debye mass and the free energy in QED by systematically expanding the sum-integrals in powers of  $e^2$  and  $m/T$ . The results are accurate to order  $e^5$ . In the two-loop  $\Phi$ -derivable approximation, both the thermodynamic potential and gap equation are finite, and so there is no running of the coupling. The solution to the gap equation is trivial; the Debye mass is given by its weak-coupling expression. This was also the case in the two-loop calculation of Blaizot,

Iancu and Rebhan in the case of QCD [30, 31, 32]. In scalar field theory, the coupling is running incorrectly by a factor of three [30, 31, 32, 34]. In the three-loop  $\Phi$ -derivable approximation, the expressions for the Debye mass and the free energy require wave-function renormalization and renormalization of the coupling constant, respectively. The running of the resulting renormalized coupling agrees with the standard one-loop running in QED.

We have also considered the problem of gauge dependence within the 2PI effective action formalism. We gave an explicit example of how gauge dependence arises in the one-loop gap equation for the photon propagator when one truncates the gap equations at order  $e^4$ . Our calculation is in agreement with general results on the gauge dependence of  $\Phi$ -derivable approximations [37].

The method could also be used to solve the three-loop  $\Phi$ -derivable approximation in QCD with an accuracy of order  $g^5$ ; however, given that the final three-loop  $\Phi$ -derivable result in QED does not seem to dramatically improve the scale variation at large coupling it is questionable whether this would be a worthwhile endeavor. Going beyond three-loops in scalar theory as well as gauge theories would be very difficult. One problem is that there are new four-loop sum-integrals of order  $g^6$  that have not yet been evaluated. In nonabelian gauge theories, there is the additional problem that the free energy at this order is sensitive to the nonperturbative momentum scale  $g^2 T$  which is associated with screening of static magnetic fields. This may cause the expansion of the three-loop  $\Phi$ -derivable approximation to break down beyond order  $g^5$ .

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## APPENDIX A: SUM-INTEGRALS

In the imaginary-time formalism for thermal field theory, the 4-momentum  $P = (P_0, \mathbf{p})$  is Euclidean with  $P^2 = P_0^2 + \mathbf{p}^2$ . The Euclidean energy  $p_0$  has discrete values:  $P_0 = 2n\pi T$  for bosons and  $P_0 = (2n + 1)\pi T$  for fermions, where  $n$  is an integer. Loop diagrams involve sums over  $P_0$  and integrals over  $\mathbf{p}$ . With dimensional regularization, the integral is generalized to  $d = 3 - 2\epsilon$  spatial dimensions. We define the dimensionally regularized sum-integral by

$$\oint_P \equiv \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon T \sum_{P_0=2n\pi T} \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \quad (A1)$$



$$\oint_{\{P\}} \equiv \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon T \sum_{P_0=(2n+1)\pi T} \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \quad (\text{A2})$$

where  $3 - 2\epsilon$  is the dimension of space and  $\mu$  is an arbitrary momentum scale. The factor  $(e^\gamma/4\pi)^\epsilon$  is introduced so that, after minimal subtraction of the poles in  $\epsilon$  due to ultraviolet divergences,  $\mu$  coincides with the renormalization scale of the  $\overline{\text{MS}}$  renormalization scheme.

### 1. One-loop sum-integrals

The specific one-loop sum-integrals needed are

$$\oint_P \log P^2 = -\frac{\pi^2 T^4}{45}, \quad (\text{A3})$$

$$\oint_P \frac{1}{P^2} = \frac{T^2}{12} \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \left[ 1 + \left( 2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \epsilon \right], \quad (\text{A4})$$

$$\oint_P \frac{1}{(P^2)^2} = \frac{1}{(4\pi)^2} \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 2\gamma \right], \quad (\text{A5})$$

$$\oint_{\{P\}} \log P^2 = \frac{7\pi^2 T^4}{360}, \quad (\text{A6})$$

$$\oint_{\{P\}} \frac{1}{P^2} = -\frac{T^2}{24} \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \left[ 1 + \left( 2 - 2 \log 2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) \epsilon \right], \quad (\text{A7})$$

$$\oint_{\{P\}} \frac{1}{(P^2)^2} = \frac{1}{(4\pi)^2} \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 2\gamma + 4 \log 2 \right]. \quad (\text{A8})$$

The errors are all one order higher in  $\epsilon$  than the smallest term shown. The calculations of these sum-integral is standard.

### 2. Two-loop sum-integrals

The two-loop sum-integrals that are needed all vanish:

$$\oint_{\{PQ\}} \frac{1}{P^2 Q^2 (P+Q)^2} = 0, \quad (\text{A9})$$

$$\oint_{PQ} \frac{1}{P^2 Q^2 (P+Q)^2} = 0, \quad (\text{A10})$$

The errors are all of order  $\epsilon$ . Details of the calculation of these two-loop sum-integrals can be found in e.g. Ref. [2].

### 3. Three-loop sum-integrals

The three-loop diagrams needed are

$$\oint_{PQR} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} = \frac{1}{(4\pi)^2} \left( \frac{T^2}{12} \right)^2 \left( \frac{\mu}{4\pi T} \right)^{6\epsilon} \left[ \frac{6}{\epsilon} + \frac{182}{5} - 12 \frac{\zeta'(-3)}{\zeta(-3)} + 48 \frac{\zeta'(-1)}{\zeta(-1)} \right], \quad (\text{A11})$$

$$\oint_{\{PQR\}} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} = \frac{1}{(4\pi)^2} \left( \frac{T^2}{12} \right)^2 \left( \frac{\mu}{4\pi T} \right)^{6\epsilon} \left[ \frac{3}{2\epsilon} + \frac{173}{20} - \frac{63}{5} \log 2 - 3 \frac{\zeta'(-3)}{\zeta(-3)} + 12 \frac{\zeta'(-1)}{\zeta(-1)} \right], \quad (\text{A12})$$

$$\oint_{PQ\{R\}} \frac{1}{P^2 Q^2 R^2 (P+Q+R)^2} = \frac{1}{(4\pi)^2} \left( \frac{T^2}{12} \right)^2 \left( \frac{\mu}{4\pi T} \right)^{6\epsilon} \left[ -\frac{3}{4\epsilon} - \frac{179}{40} + \frac{51}{10} \log 2 + \frac{3}{2} \frac{\zeta'(-3)}{\zeta(-3)} - 6 \frac{\zeta'(-1)}{\zeta(-1)} \right], \quad (\text{A13})$$

$$\oint_{\{P\}QR} \frac{QR}{P^2 Q^2 R^2 (P+Q)^2 (P+R)^2} = \frac{1}{(4\pi)^2} \left( \frac{T^2}{12} \right)^2 \left( \frac{\mu}{4\pi T} \right)^{6\epsilon} \left[ \frac{3}{8\epsilon} + \frac{9}{4} \gamma + \frac{361}{160} + \frac{57}{10} \log 2 + \frac{3}{2} \frac{\zeta'(-3)}{\zeta(-3)} - \frac{3}{2} \frac{\zeta'(-1)}{\zeta(-1)} \right], \quad (\text{A14})$$

$$\oint_{P\{QR\}} \frac{(QR)^2}{P^2 Q^2 R^2 (P+Q)^2 (P+R)^2} = \frac{1}{(4\pi)^2} \left( \frac{T^2}{12} \right)^2 \left( \frac{\mu}{4\pi T} \right)^{6\epsilon} \left[ \frac{5}{24\epsilon} + \frac{1}{4} \gamma + \frac{23}{24} - \frac{8}{5} \log 2 - \frac{1}{6} \frac{\zeta'(-3)}{\zeta(-3)} + \frac{7}{6} \frac{\zeta'(-1)}{\zeta(-1)} \right]. \quad (\text{A15})$$

The errors are all of order  $\epsilon$ . The calculation of these three-loop sum-integrals was done in Ref. [2] and details can be found there.

### APPENDIX B: INTEGRALS

Dimensional regularization can be used to regularize both the ultraviolet divergences and infrared divergences in 3-dimensional integrals over momenta. The spatial dimension is generalized to  $d = 3 - 2\epsilon$  dimensions. Integrals are evaluated at a value of  $d$  for which they converge and then analytically continued to  $d = 3$ . We use the inte-

gration measure

$$\int_p \equiv \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}. \quad (\text{B1})$$

We require a few integrals in that appear in the soft sector. The momentum scale in these integrals is set by the Debye mass  $m$ . The one-loop integrals needed are:

$$\int_p \log(p^2 + m^2) = -\frac{m^3}{6\pi}, \quad (\text{B2})$$

The errors are all of order  $\epsilon$ . The calculation of these integrals is standard.

$$\int_p \frac{1}{p^2 + m^2} = -\frac{m}{4\pi}. \quad (\text{B3})$$

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